Simple $\pi\pi$ Self-Consistent Calculation Using the Strip Approximation*

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Using a simple representation derived from the strip approximation by Singh and Udgaonkar (and resembling the Cini-Fubini approximation), an integral equation can be obtained for the partial-wave amplitude. This can be solved by the N/D method. By making certain approximations, the resulting equations, when combined with the requirement of self-consistency, lead to simple algebraic equations for determining the parameters of the ρ resonance. The only parameter is the width of the strip. This, however, can be calculated by using the Chew-Frautschi principle of maximum strength.

1. INTRODUCTION

SINCE Chew and Mandelstam[1] first introduced the bootstrap idea, many calculations have been attempted in which particle parameters have been calculated self-consistently. Most of the methods used are quite difficult to implement in practice, however. In particular, they often entail the numerical evaluation of integrals, which usually requires the aid of an electronic computer. The difficulties become particularly acute if one attempts multichannel calculations, which are essential for testing the approximation of keeping only the lowest intermediate states in the unitarity condition. It would thus be desirable to find an approximate approach whose principal feature is the simplicity of the numerical calculations. This would make it possible to extend bootstrap calculations to much more complicated problems.

In the present approach, we shall develop a method which is particularly amenable to simplifying approximations. We start from the strip approximation of Chew and Frautschi.[2] Although the same final result can often be obtained by starting from a partial-wave dispersion relation with a cutoff, the present approach is valid even if distant singularities are important. The usual partial-wave dispersion relation arguments would fail in this case.

In Sec. 2, the strip approximation is used to obtain a representation for the partial-wave amplitude, following a procedure suggested by Singh and Udgaonkar.[3] This representation resembles the Cini-Fubini approximation.[4] By imposing unitarity, we obtain a nonlinear integral equation. In Sec. 3, this is reduced to linear equations by the N/D method. These equations can be solved by using a pole approximation.

In Sec. 4, the general method is applied to a self-consistent calculation of the ρ resonance. By making certain reasonable approximations, the entire calculation reduces to the solution of a quadratic equation in the resonance position. The only remaining parameter is the width of the strip. In Sec. 5, methods of calculating this parameter are discussed. Also, an actual calculation with somewhat improved approximations is carried out, in which the principle of maximum strength[2] is imposed. This leads to somewhat more complicated equations which, however, are still algebraic and can be easily solved to obtain the strip width as well as the ρ parameters. These results are completely independent of any experimental quantity except the pion mass, which is needed to fix the energy scale.

2. THE SINGH-UDGAONKAR REPRESENTATION

For simplicity, we shall consider the scattering of two spinless equal-mass particles, although a similar procedure can be followed for other scattering problems. If we set the mass $\mu=1$, the Mandelstam representation[5] gives

$$A(s,t)=\frac{1}{\pi}\int_4^\infty dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_4^\infty du'\frac{A_u(u',s)}{u'-u}, \quad (1)$$

where $A(s,t)$ = invariant amplitude, s = (total energy)^2, $t=-2\nu(1-\cos\theta)$, $\nu=(s/4)-1$, θ = scattering angle, $u=4-s-t$, $A_t=t$-channel absorptive part, and $A_u=u$-channel absorptive part. The integrals in Eq. (1) can be defined either in the elementary sense or by continuation.[6] We shall find it convenient to split each integral into two portions to give

$$A(s,t)=\frac{1}{\pi}\int_4^{t_1} dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_4^{u_1} du'\frac{A_u(u',s)}{u'-u}+\left[\frac{1}{\pi}\int_{t_1}^\infty dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_{u_1}^\infty du'\frac{A_u(u',s)}{u'-u}\right], \quad (2)$$

where t_1 and u_1 are the values which separate the low- and high-energy regions in the crossed channels. The regions $t<t_1$ and $u<u_1$ are essentially the resonance regions where two-body unitarity is roughly valid, while $t>t_1$ and $u>u_1$ are the high-energy continuum regions. In other words, t_1 and u_1 correspond to strip widths in a strip approximation.[2]

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[1] G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).
[2] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961).
[3] V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963).
[4] M. Cini and S. Fubini, Ann. Phys. (Paris) **3**, 585 (1960).
[5] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[6] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

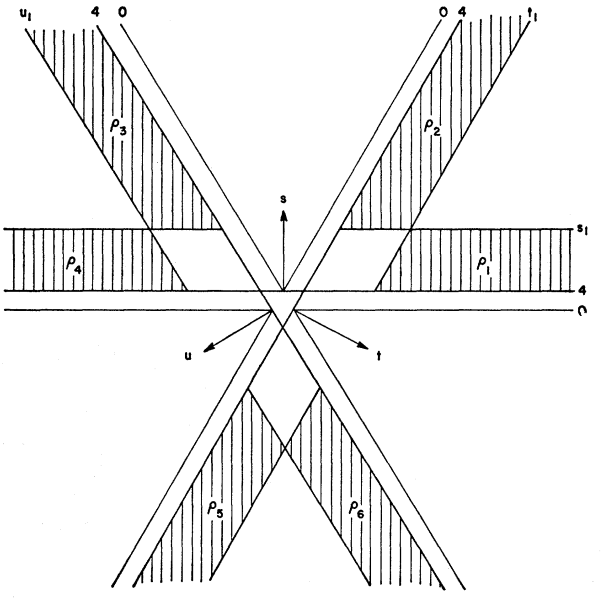

FIG. 1. The Mandelstam diagram. The shaded areas are assumed to dominate.

In what follows, partial-wave expansions will be made for A_t and A_u at low energies in the crossed channels, i.e., for $t<t_1$ and $u<u_1$. This can be used to evaluate the first two integrals in Eq. (2). To evaluate the last two integrals, we shall now follow the procedure of Singh and Udgaonkar.[3] This is physically similar to the Cini-Fubini procedure, although differing from it in several important details. The starting point is the strip approximation, according to which the double spectral functions are negligible everywhere except in the shaded regions of Fig. 1. In other words, for high s, for instance, only the low t contribution is important. This corresponds to long-range forces dominating at high energies. Such a "peripheral" approximation seems to be quite reasonable[2] and is the basis for most theories of high-energy scattering.

Suppose that, from now on, we restrict ourselves to $0<s<s_1$. Then in the regions $t>t_1$ and $u>u_1$, the effect of ρ_5 and ρ_6 can be safely dropped, since they are quite distant singularities. In these regions, the strip approximation thus gives

$$A_t(t,s)=\frac{1}{\pi}\int_4^{s_1} ds'\frac{\rho_1(s',t)}{s'-s}, \tag{3}$$

$$A_u(u,s)=\frac{1}{\pi}\int_4^{s_1} ds'\frac{\rho_4(s',u)}{s'-s}. \tag{4}$$

But for $4<s<s_1$, the s-channel absorptive part A_s is given by

$$A_s(s,t)=\frac{1}{\pi}\int_{t_1}^{\infty} dt'\frac{\rho_1(s,t')}{t'-t}+\frac{1}{\pi}\int_{u_1}^{\infty} du'\frac{\rho_4(s,u')}{u'-u}. \tag{5}$$

Interchanging the order of integration and using Eqs. (3) and (4), we therefore get

$$\frac{1}{\pi}\int_4^{s_1} ds'\frac{A_s(s',t)}{s'-s}=\frac{1}{\pi}\int_{t_1}^{\infty} dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_{u_1}^{\infty} du'\frac{A_u(u',s)}{u'-u}. \tag{6}$$

Equation (2) now becomes

$$A(s,t)=\frac{1}{\pi}\int_4^{s_1} ds'\frac{A_s(s',t)}{s'-s}+\frac{1}{\pi}\int_4^{t_1} dt'\frac{A_t(t',s)}{t'-t}+\frac{1}{\pi}\int_4^{u_1} du'\frac{A_u(u',s)}{u'-u}. \tag{7}$$

Suppose we make a partial-wave expansion

$$A_s(s,t)=\sum(2l+1)\,\mathrm{Im}A_l(\nu)P_l(1+t/2\nu), \tag{8}$$

and similarly for A_t and A_u. From Fig. 1 it is obvious that these expansions will converge within the integrals of Eq. (7), provided we restrict $A(s,t)$ to the $s<s_1$ part of the physical region. Thus, we can express $A(s,t)$ entirely in terms of partial-wave amplitudes. If we now substitute Eq. (8) into Eq. (7) and project out the lth partial wave using

$$A_l(\nu)=\frac{1}{2}\int_{-1}^{1} d(\cos\theta)P_l(\cos\theta)A(s,t), \tag{9}$$

we obtain

$$A_l(\nu)=\frac{\nu^l}{\pi}\int_0^{\nu_1} d\nu'\frac{\mathrm{Im}A_l(\nu')}{\nu'^l(\nu'-\nu)}+F_l(\nu), \tag{10}$$

where

$$F_l(\nu)=V_l(\nu)+(\text{contribution of waves}>l), \tag{11}$$

$$V_l(\nu)=\frac{1}{2\pi\nu}\int_4^{t_1} dt'A_t(t',\,4\nu+4)Q_l\left(1+\frac{t'}{2\nu}\right)+\frac{(-1)^l}{2\pi\nu}\int_4^{u_1} du'A_u(u',\,4\nu+4)Q_l\left(1+\frac{u'}{2\nu}\right), \tag{12}$$

and $\nu_1=(s_1/4)-1$. Equation (10) resembles the usual partial-wave dispersion relation, except for the second term on the right side of Eq. (11), which is a polynomial in ν. Unless waves $>l$ are small, this term can be quite important. Moreover, we have a cutoff ν_1 which arises naturally, and is in fact needed to prevent effects within the strips from being counted twice. Also, as we have seen, we are permitted to use partial-wave expansions for A_t and A_u to evaluate $V_l(\nu)$. This we cannot do with the usual partial-wave dispersion relation unless nearby singularities dominate, a situation which does not usually prevail.

3. THE N/D EQUATIONS

If we impose the elastic unitarity condition

$$\mathrm{Im}A_l(\nu)=[\nu/(\nu+1)]^{1/2}|A_l(\nu)|^2, \tag{13}$$

Eq. (10) becomes a nonlinear integral equation. It is interesting to note that we need Eq. (13) only in the region $0<\nu<\nu_1$. This is because the region $\nu>\nu_1$ has already been taken into account through the strip approximation, which means that a large class of inelastic effects is automatically included. This should be contrasted with most methods, where such effects have to be explicitly put in. Of course, this is true only for the region $\nu>\nu_1$. Inelastic effects for $\nu<\nu_1$ would have to be inserted if accurate results are desired.

We can solve Eq. (10) in the usual way by the N/D method. The Uretsky version of this method[7] gives

$$A_l(\nu)=N(\nu)/D(\nu), \tag{14}$$

with

$$N(\nu)=F_l(\nu)D(\nu)$$
$$+\frac{\nu^l}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{F_l(\nu')N(\nu')}{\nu'^l(\nu'-\nu)}, \tag{15}$$

and

$$D(\nu)=1-\frac{1}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{N(\nu')}{\nu'-\nu}, \tag{16}$$

where we have normalized D to unity at infinity. It is a straightforward matter to verify that this N/D representation satisfies unitarity in the interval $0<\nu<\nu_1$ and has the same singularities as Eq. (10) elsewhere.[8] One approximate way of solving Eqs. (15) and (16) is to use the pole method. We make the approximation

$$F_l(\nu)\simeq\nu^l\sum_{i=1}^{n}\frac{b_i}{\omega_i+\nu} \tag{17}$$

in the region $0<\nu<\nu_1$. The equations can then be solved exactly to give

$$N(\nu)=\nu^l\sum_{i=1}^{n}\frac{b_i}{\omega_i+\nu}D(-\omega_i) \tag{18}$$

and

$$D(\nu)=1-\sum_{i=1}^{n}b_iH(\nu,\omega_i)D(-\omega_i), \tag{19}$$

with

$$H(\nu,\omega_i)=\frac{1}{\pi}\int_0^{\nu_1}d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{\nu'^l}{(\nu'+\omega_i)(\nu'-\nu)}. \tag{20}$$

This result can be checked by substituting into Eqs. (15) and (16) and seeing that they are satisfied. To determine the $D(-\omega_i)$, we simply evaluate $D(\nu)$ at $\nu=-\omega_j$ to give

$$D(-\omega_j)=1-\sum_{i=1}^{n}b_iH(-\omega_j,\omega_i)D(-\omega_i). \tag{21}$$

If we set $j=1,\cdots,n$, Eq. (21) represents a set of n simultaneous linear equations in the $D(-\omega_i)$.

4. THE ρ BOOTSTRAP

We shall now specialize to the $\pi\pi$ problem, assuming that it is dominated by the $I=1$, $l=1$, ρ resonance at low energies.[9] In this problem,[10] Bose statistics imply that all three channels are symmetric and that $A_u(t,s)=(-1)^IA_t(t,s)=(-1)^lA_t(t,s)$. If we use the partial-wave expansion (8) in the t and u channels, Eq. (12) thus becomes

$$V_l(\nu)=\frac{3\beta_{I1}}{\pi\nu}\int_4^{t_1}dt'$$
$$\times\mathrm{Im}A_1\left(\frac{t'}{4}-1\right)P_1\left(1+8\frac{\nu+1}{t'-4}\right)Q_l\left(1+\frac{t'}{2\nu}\right). \tag{22}$$

Here, β_{I1} is the crossing matrix element connecting the $I=1$ state in the t channel to a state with isotopic spin I in the s channel. In our case, $\beta_{01}=1$, $\beta_{11}=\frac{1}{2}$, and $\beta_{21}=-\frac{1}{2}$.

Suppose we approximate $F_l(\nu)$ in the $I=1$, $l=1$ state by its threshold behavior. Then, assuming that higher waves are unimportant and using Eq. (22),

$$F_1(\nu)\simeq V_1(\nu)\simeq\nu b, \tag{23}$$

with

$$b=\frac{\beta_{11}}{\pi}\int_0^{\nu_1}d\nu'\frac{\nu'+2}{\nu'(\nu'+1)^2}\mathrm{Im}A_1(\nu'), \tag{24}$$

where $\nu'=(t'/4)-1$ and $\nu_1=(t_1/4)-1=(s_1/4)-1$. This threshold approximation can always be justified *a posteriori* (see Fig. 2). Mathematically, it just corresponds to the pole approximation of the preceding section with $n=1$, $b=(b_1/\omega_1)$, and $\omega_1=\infty$. Since D is normalized to unity at $\nu=-\infty$, we thus have

$$N(\nu)=\nu b \tag{25}$$

and

$$\mathrm{Re}D(\nu)=1-bh(\nu), \tag{26}$$

where

$$h(\nu)=\lim_{\omega_1\to\infty}\omega_1\,\mathrm{Re}H(\nu,\omega_1).$$

If we make the further approximation that $h(\nu)$ is linear, with the exact value and derivative at threshold,

[7] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[8] Strictly speaking, this is true only if $F_l(\nu)$ is the integral over all singularities of the exact $A_l(\nu)$ outside the region $0<\nu<\nu_1$. From Eq. (10), however, we see that we can always replace this function by the approximation (11) in the interval $0<\nu<\nu_1$, which is the only region where $F_l(\nu)$ is needed anyway.

[9] G. Button, G. R. Kalbfleisch, G. R. Lynch, B. C. Maglić, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. **126**, 1858 (1962).

[10] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

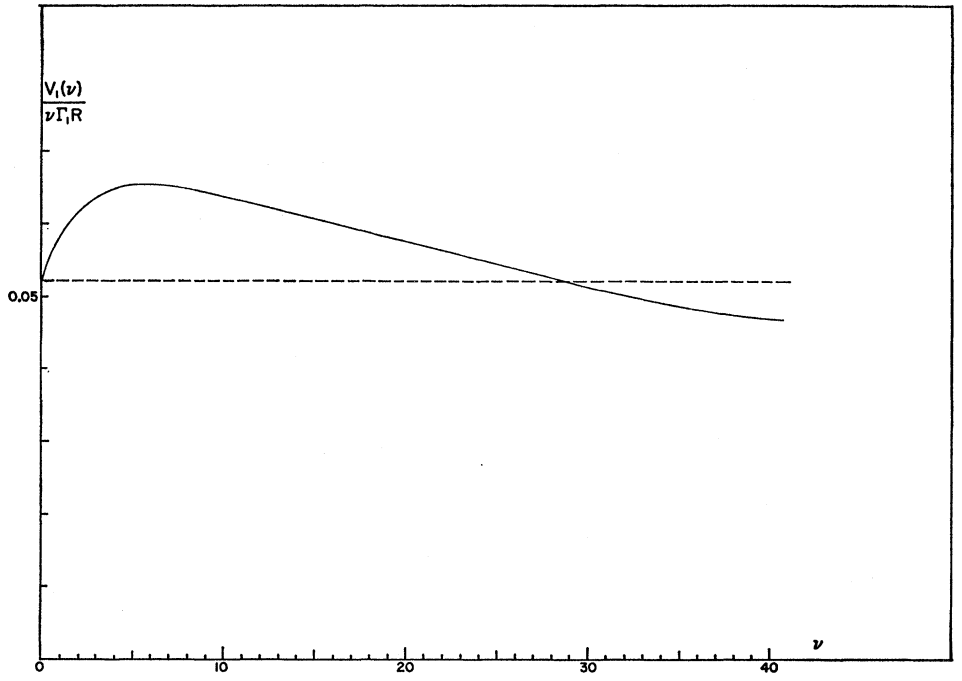

FIG. 2. Typical plot of $V_1(\nu)/(\nu\Gamma_1 R)$ using Eqs. (22) and (34), with $\nu_R=3$. The error in making the threshold approximation (23) (dashed line) is seen to be of the order of 15%.

and use the fact that $\nu_1 \gg 1$, we obtain

$$h(\nu)=\frac{1}{\pi}-[(\nu_1-\tfrac{1}{2}\ln 4\nu_1)+\nu\ln 4\nu_1]. \tag{27}$$

Suppose we have a resonance at $\nu=\nu_R$. Then

$$\mathrm{Re}D(\nu_R)=1-bh(\nu_R)=0, \tag{28}$$

and the half-width in the ν variable is

$$\Gamma_1=[-N(\nu_R)/\mathrm{Re}D'(\nu_R)]=[\pi\nu_R/\ln 4\nu_1]. \tag{29}$$

We can thus find Γ_1 once we know ν_1 and ν_R. To find ν_R from Eq. (28), we must evaluate b. Now from Eq. (16), $\mathrm{Im}D=-[\nu/(\nu+1)]^{1/2}N$. Using this together with Eqs. (14), (25), (26), (27), and (29), we obtain

$$\mathrm{Im}A_1(\nu)=\frac{[(\nu+1)/\nu]^{1/2}(\nu/\nu_R)^2\Gamma_1^2}{(\nu-\nu_R)^2+[\nu/(\nu+1)](\nu/\nu_R)^2\Gamma_1^2}. \tag{30}$$

Let us insert Eq. (30) into Eq. (24). If we assume that the ρ resonance is narrow, we can neglect all ν' dependence in the integrand except for the term $(\nu'-\nu_R)^2$. Everywhere else, we shall set $\nu'=\nu_R$. We then have

$$b=[\beta_{11}\Gamma_1 R(\nu_R+2)]/[\nu_R(\nu_R+1)]^2, \tag{31}$$

where

$$R=(1/\pi)[\tan^{-1}((\nu_1-\nu_R)/\nu_R)a+\tan^{-1}a], \tag{32}$$

and

$$a=(\nu_R/\Gamma_1)[(\nu_R+1)/\nu_R]^{1/2}. \tag{33}$$

Equation (31) is equivalent to inserting the delta function approximation

$$\mathrm{Im}A_1(\nu)=\pi\Gamma_1 R\delta(\nu-\nu_R) \tag{34}$$

into Eq. (24). If the resonance is narrow, a is large, in which case $R\simeq 1$. The approximation is then equivalent to treating the resonance as a stable particle.

Therefore, taking $R=1$, and inserting Eq. (31) into Eq. (28), we have, using Eq. (27),

$$\mathrm{Re}D(\nu_R)=1-\beta_{11}\frac{\nu_R+2}{(\nu_R+1)^2}\left[\frac{\nu_1}{\ln 4\nu_1}-\frac{1}{2}+\nu_R\right]=0. \tag{35}$$

This is just a trivial quadratic equation for ν_R, if we take the strip width ν_1 to be given. Once ν_R has been determined from Eq. (35), Γ_1 can be found immediately from Eq. (29). Some of the results are shown in Table I. In choosing ν_1, we have assumed that the strip width corresponds roughly to the resonance region. Since the highest known resonance in the $\pi\pi$ system is the f^0 with $\nu\simeq 20$,[11] we thus take $\nu_1=20$. The case $\nu_1=40$ is given to show the sensitivity to ν_1. From Fig. 1 it is obvious, however, that this sensitivity is reduced if a more accurate approximation than Eq. (23) were made for $F_1(\nu)$.

In both of the cases shown in Table I, the second solution of the quadratic equation (35) has $\nu_R<-1$. This is unphysical and is thus rejected.

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[11] W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 272 (1962); J. J. Veillet, G. Hennessy, H. Bingham, M. Bloch, D. Drijard *et al.*, *ibid.* **10**, 29 (1963).

5. DETERMINATION OF THE STRIP WIDTH

In the above method, the strip widths are arbitrary parameters. Suppose that in a complete scheme, where all particle masses are calculated, we set all strip widths equal. Then as pointed out by Chew,[12] this width w can be used to fix the energy scale, which in any case is undetermined, since only energy ratios are calculable in a bootstrap scheme. For instance, if the pion mass is taken to be unity, we can vary w until the mass of the pion calculated in, say, the $\pi\rho$ problem is also unity. This determines the value of w.

In a more limited calculation, such as the ρ bootstrap of Sec. 4, we cannot calculate w by this method. However, it was suggested by Chew[13] that w could be fixed by imposing the principle of maximum strength.[2] In the simple version of the $\pi\pi$ problem given in Sec. 4, this means that if we continue the N/D equations together with Eq. (22) to unphysical values of l, then in the unphysical "state" $I=0$, $l=1$, we must have $D(-1)=0$. In other words, this "state" must have a bound state at $s=0$. This guarantees that the Pomeranchuk Regge trajectory is such that the total cross section is constant at very high energies.

If we attempt to fix ν_1 in the calculation of Sec. 4 by this method, we find that the approximation $R=1$ leads to difficulties. We shall thus use its actual value as given by Eqs. (32) and (33). This is just a crude way of bringing in some of the effects of the finite width of the ρ.[14] Equation (35) is then modified to

$$\mathrm{Re}D(\nu_R)=1-\beta_{11}R\frac{\nu_R+2}{(\nu_R+1)^2}\left[\frac{\nu_1}{\ln 4\nu_1}-\frac{1}{2}+\nu_R\right]=0. \quad (36)$$

Similarly, the condition that $\mathrm{Re}D(-1)=0$ in the $I=0$, $l=1$ "state" leads to

$$\mathrm{Re}D(-1)=1-\beta_{01}R\frac{\nu_R+2}{(\nu_R+1)^2}\left[\frac{\nu_1}{\ln 4\nu_1}-\frac{3}{2}\right]=0. \quad (37)$$

Equations (36), (37), and (32) can now be solved for R, ν_R, and ν_1. Thus, the quantity a in Eq. (32) can be found by using Eqs. (29) and (33), which give

$$a=(\ln 4\nu_1/\pi)[(\nu_R+1)/\nu_R]^{1/2}. \quad (38)$$

To obtain ν_R, we first write Eq. (37) in the form

$$\nu_1/\ln 4\nu_1=\tfrac{3}{2}+[(\nu_R+1)^2/R(\nu_R+2)]. \quad (39)$$

If we substitute this into Eq. (36), we obtain

$$R=(\nu_R+1)/(\nu_R+2), \quad (40)$$

which, when inserted into Eq. (39), gives

$$\nu_1/\ln 4\nu_1=\nu_R+\tfrac{5}{2}. \quad (41)$$

If we now choose a particular value of ν_1, we can obtain ν_R from Eq. (41). Then R can be calculated both from Eq. (40) and from Eqs. (38) and (32). The self-consistent value is just the point at which these two curves of R versus ν_1 cross. This gives $\nu_1=26$, where $R=0.804$. At this point, $\nu_R=3.1$ and Eq. (29) gives $\Gamma_1=2.1$. This corresponds to a mass and half-width for the ρ of 560 MeV and 126 MeV, respectively.

TABLE I. Self-consistent values of the ρ parameters for given strip widths. The experimental mass and width (Ref. 9) are 767 and 120 MeV, respectively.

R	ν_1	ν_R	Γ_1	Mass (MeV)	Width (MeV)
1	20	3.7	2.6	600	300
	40	7.3	4.5	800	410
0.804	26	3.1	2.1	560	250

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[12] G. F. Chew (unpublished).
[13] G. F. Chew, Phys. Rev. **129**, 2363 (1963).
[14] Thus, in the calculation of this section, it is found *a posteriori* that the b calculated from Eq. (31) differs from the correct value calculated from Eqs. (24), (25), and (26) by about 30%. On the other hand, the approximation $R=1$ increases the error to about 60%. Nevertheless, for a given ν_1, this approximation does not seem to change the self-consistent results in any drastic way, as can be seen from Table I.